4. Group actions

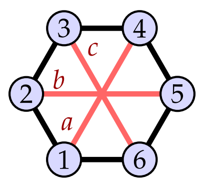
"Groups, as men, will be known by their actions."
- Guillermo Moreno

4.1. **The definition.**

Definition 4.1. An ***action*** of a group G on a set X is a function from $\alpha : G \times X \to X$ such that the following hold for all $g, h \in G$ and all $x \in X$; we write $g \cdot x$ in place of $\alpha(g, x)$.

(1) $g \cdot (h \cdot x) = (gh) \cdot x$, and
(2) $1 \cdot x = x$.

Problem 4.2. Recall that D_6 is the automorphism group of the regular hexagon $\mathcal{D}_6$. Let V be the set of vertices of $\mathcal{D}_6$, let E the set of edges of $\mathcal{D}_6$, and let $X = \{a, b, c\}$ be the set of (three) diagonal edges shown below.

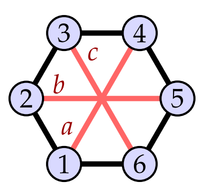


(1) Show that D_6 acts on V via the rule $\sigma \cdot v = \sigma(v)$ for all $\sigma \in D_6$ and all $v \in V$.
(2) Show that D_6 acts on E via the rule $\sigma \cdot (v_1, v_2) = (\sigma(v_1), \sigma(v_1))$ for all $\sigma \in D_6$ and all $(v_1, v_2) \in E$.
(3) Show that D_6 acts on X via the rule $\sigma \cdot (v_1, v_2) = (\sigma(v_1), \sigma(v_1))$ for all $\sigma \in D_6$ and all $(v_1, v_2) \in X$.

Definition 4.3. Let G act on X.

(1) The action is ***transitive*** if for every $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$.
(2) For $g \in G$ and $x \in X$, we say that g ***fixes*** x if $g \cdot x = x$.
(3) For $x \in X$, the ***stabilizer of*** x, denoted G_x, is set of all $g \in G$ that fix x.

Problem 4.4. Let $G = D_6$, and consider the action of G on $X = \{a, b, c\}$ defined by the rule $\sigma \cdot (v_1, v_2) = (\sigma(v_1), \sigma(v_1))$.



(1) Is the action transitive?
(2) Determine the G_a.
(3) Find a numerical relationship between $|G|$, $|X|$, and $|G_a|$.
(4) Determine $G_{a,b}$ where $G_{a,b}$ is the set of elements of G that fix both a and b.
(5) Are there elements of G that fix *every* element of X? If so, find them all.

Theorem 4.5. *An action of a group G on X is transitive if there exists some $x \in X$ such that for all $y \in X$ there is a $g \in G$ for which $g \cdot x = y$.*

Definition 4.6. Let G act on X.

(1) The ***kernel*** of the action is the subset of G that fixes every $x \in X$.
(2) The action is said to be ***faithful*** if the kernel is trivial.

Theorem 4.7. *If G acts on X and $x \in X$, then G_x is a subgroup of G, and the kernel of the action is a normal subgroup.*

Theorem 4.8. *Let G act on X. For every $g \in G$, define $\sigma_g : X \to X$ by $\sigma_g(x) = g \cdot x$. Then σ_g is a bijection, i.e. $\sigma_g \in \mathrm{Sym}(X)$.* [Hint: make use of the fact that g has an inverse.]

Theorem 4.9. *Let G act on X, and define $\sigma : G \to \mathrm{Sym}(X)$ by $\sigma(g) = \sigma_g$ where σ_g is defined as in the previous theorem. Then σ is a homomorphism.* [Hint: in order to show that $\sigma_{gh} = \sigma_g \circ \sigma_h$, show that $\sigma_{gh}(x) = \sigma_g(\sigma_h(x))$ for all $x \in X$.]

Definition 4.10. In the previous theorem, the function $\sigma : G \to \mathrm{Sym}(X)$ is called the ***associated permutation representation*** of the action of G on X.

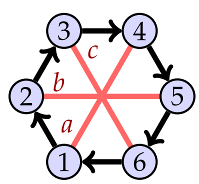
Remark 4.11. Observe that the kernel of an action corresponds with the kernel of the associated permutation representation, so an action is faithful if and only if the associated permutation representation is injective.

Problem 4.12. As in Problem 4.4, consider the action of D_6 the 3 diagonals of $\mathcal{D}_6$.

(1) What is the kernel of the action? Is the action faithful?
(2) What is the image of the associated permutation representation?

Note: the kernel is a subgroup of D_6; the image of the representation is a subgroup of $\mathrm{Sym}(a, b, c)$.

Problem 4.13. Let $G = C_6$. As with D_6, we have an action of G on $X = \{a, b, c\}$ defined by $\sigma \cdot (v_1, v_2) = (\sigma(v_1), \sigma(v_1))$.



(1) Is the action transitive?
(2) Determine the G_a.
(3) Find a numerical relationship between $|G|$, $|X|$, and $|G_a|$.
(4) What is the kernel of the action. Is the action faithful?
(5) What is the image of the associated permutation representation?

4.2. Action by left multiplication.

Theorem 4.14 (Action by left multiplication)**.** *Let G be a group, and let H be a subgroup. Then the rule $g \cdot aH = (ga)H$ defines an action of G on the coset space G/H.*

Problem 4.15. Let G be a group, and let H be a subgroup. Consider the action of G on G/H by left multiplication (as in the previous theorem).

(1) Is the action transitive?
(2) Show that the stabilizer of the coset aH is aHa^{-1}.
(3) Show that the kernel of the action is $\bigcap_{a \in G} aHa^{-1}$. (Note that, in particular, this shows that the kernel is a normal subgroup of G contained in H.)
(4) Give an example of a group G and a proper nontrivial subgroup H for which this action is **not** faithful.

Definition 4.16. A group is ***simple*** if it has *no* proper nontrivial normal subgroups.

Remark 4.17. Whenever a group G has a proper nontrivial normal subgroup N, we can break G into two "simpler" pieces: N and G/N. The simple groups are the groups that can not be broken down this way; they may be thought of as the building blocks of all groups.

Theorem 4.18. *If G is an infinite group with a proper subgroup H of finite index, then G is **not** simple.* [Hint: argue by contradiction, and consider the action of G on G/H by left multiplication. This gives rise to the associated representation $\sigma : G \to \mathrm{Sym}(G/H)$. Now, if G is simple, what do you know about the kernel of the action? What does the First Isomorphism Theorem, i.e. Theorem 3.78 and Remark 3.79, tell you?]

Theorem 4.19. *Let G be a finite group with a proper subgroup H, and let $n = |G : H|$. If $|G|$ does not divide $n!$, then G is **not** simple.* [Hint: same hint as the previous problem.]

4.3. **Action by conjugation.**

Notation 4.20. Let G be a group. For $g \in G$, the function $\gamma_g : G \to G$ defined by $\gamma_g(h) = ghg^{-1}$ is called ***conjugation by*** g.

Theorem 4.21. *If G is a group and $g \in G$, then γ_g is an automorphism of G. In particular,*
(1) if $h \in G$, then $|h| = |ghg^{-1}|$, and
(2) if H is a subgroup of G, then gHg^{-1} is a subgroup of G with $H \cong gHg^{-1}$.

Theorem 4.22. *Let $\sigma, \tau \in S_n$. If the disjoint cycle decomposition of σ is*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{k_1} \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \cdots & b_{k_2} \end{pmatrix} \cdots,$$

then the disjoint cycle decomposition of $\tau\sigma\tau^{-1}$ is

$$\begin{pmatrix} \tau(a_1) & \tau(a_2) & \cdots & \tau(a_{k_1}) \end{pmatrix} \begin{pmatrix} \tau(b_1) & \tau(b_2) & \cdots & \tau(b_{k_2}) \end{pmatrix} \cdots.$$

In particular, σ and $\tau\sigma\tau^{-1}$ have the same cycle type. [Hint: let $\psi = \tau\sigma\tau^{-1}$, and note that the theorem simply states that for all $x, y \in \{1, \ldots, n\}$ if $\sigma(x) = y$, then $\psi(\tau(x)) = \tau(y)$.]

Theorem 4.23 (Action by conjugation)**.** *Let G be a group. Then*
(1) the rule $g \cdot a = gag^{-1}$ defines an action of G on G, and
(2) the rule $g \cdot H = gHg^{-1}$ defines an action of G on the set of all subgroups of G.

Definition 4.24. Let H be a subgroup of a group G. The set $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$ is called the ***normalizer*** of H in G.

Remark 4.25. Note that we have some overlapping terminology. When G acts on itself by conjugation (as in the first part of the previous theorem), the stabilizer of an element a of G is $C_G(a)$. When G acts on its subgroups by conjugation (as in the second part of the previous theorem), the stabilizer of a subgroup H is $N_G(H)$.

Theorem 4.26. *If H is a subgroup of a group G, then H is a normal subgroup of $N_G(H)$.*

Definition 4.27. If A and B are subsets of a group G, we define $AB := \{ab | a \in A, b \in B\}$.

Theorem 4.28. *If H is a subgroup of a group G and K is a subgroup of $N_G(H)$, then KH is a subgroup of $N_G(H)$.*

Definition 4.29. Let G be a group acting on a set X. If $x \in X$, then the subset of X defined by $Gx := \{g \cdot x | g \in G\}$ is called the ***orbit of** x **under*** G.

Theorem 4.30. *If G is a group acting on a set X, then the set of orbits forms a partition of X.*

Definition 4.31. When G acts on itself (or on its subgroups) by conjugation, the orbits are called ***conjugacy classes*** (or ***conjugacy classes of subgroups***) and two elements in the same conjugacy class are said to be ***conjugate***.

Problem 4.32. Determine the conjugacy classes of S_3. Determine the conjugacy classes of subgroups of S_3.

Theorem 4.33. *Two elements of S_n are conjugate if and only if they have the same cycle type.*

Theorem 4.34. *If $n \geq 3$, then $Z(S_n) = \{1\}$.*

Problem 4.35. Determine the conjugacy classes of D_4.

4.4. **The Orbit-stabilizer Theorem.**

Remark 4.36. Suppose that G acts on X. Observe that, for any orbit $\mathcal{O}$, the action of G on X restricts to an action of G on $\mathcal{O}$, and this latter action is now transitive. In this way, many questions about group actions can be reduced to questions about transitive group actions.

Theorem 4.37 (Orbit-stabilizer Theorem)**.** *Let G be a group acting on a set X. Then for every $x \in X$, $|Gx| = |G : G_x|$.* [Hint: construct a bijection from G/G_x to Gx.]

Notation 4.38. For a group G acting on a set X, we define $\mathrm{Fix}(G)$ to be the set of all $x \in X$ such that x fixed by *every* element of G, i.e. $\mathrm{Fix}(G)$ is the set of fixed points of G. The elements of $\mathrm{Fix}(G)$ represent the orbits of G of size 1.

Theorem 4.39. *Let G be a finite group acting on a finite set X. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the orbits of G not contained in* $\mathrm{Fix}(G)$*, if any, and let $x_1, \ldots x_n \in X$ be such that $x_i \in \mathcal{O}_i$. Then*

$$|X| = |\operatorname{Fix}(G)| + \sum_{i=1}^{n} |G : G_{x_i}|.$$

[Hint: recall that the orbits of G partition X.]

Theorem 4.40. *Let P be a group of order p^k for some prime p. If P acts on a finite set X, then $|\operatorname{Fix}(P)| \equiv |X|$ modulo p.*

Theorem 4.41. *If P is a group of order p^k for some prime p, then $Z(P)$ is nontrivial.* [Hint: let P act on itself by conjugation. What is $\operatorname{Fix}(P)$ with respect to this action?]

Theorem 4.42. *If P is group of order p^2 for some prime p, then P is abelian.*

Problem 4.43 (The Class Equation)**.** Let G be a finite group. Let $C_1, \ldots, C_n$ be the conjugacy classes of G not contained in $Z(G)$, if any, and let $x_1, \ldots x_n \in G$ be such that $x_i \in C_i$. Explain how Theorem 4.39 can be used to quickly deduce that

$$|G| = |Z(G)| + \sum_{i=1}^{n} |G : C_G(x_i)|.$$

Theorem 4.44. *Let G be a finite group. If p is a prime dividing $|G|$, then p divides $|C_G(g)|$ for some nontrivial $g \in G$.* [Hint: class equation.]

Theorem 4.45 (Cauchy's Theorem)**.** *Let G be a finite group. If p is a prime dividing $|G|$, then G has an element of order p.* [Hint: consider a minimal counterexample, and first show that it must have a nontrivial center.]

Theorem 4.46. *Let p be a prime. If G is a finite group, then G is a p-group (see Definition 3.46) if an only if $|G| = p^k$ for some $k \in \mathbb{N}$.*

Problem 4.47. We should always be asking if we can generalize things. Make at least two conjectures related to generalizing (or not being able to generalize) Cauchy's Theorem.