5. Sylow's Theorem

"For a group theorist, Sylow's Theorem is such a basic tool, and so fundamental, that it is used almost without thinking, like breathing."
- Geoff Robinson

5.1. **The definition.**

Definition 5.1. Let p be a prime. A subgroup P of G is called a ***Sylow p-subgroup*** if P is a p-group and P is not properly contained in another p-subgroup of G, i.e. P is a maximal p-subgroup of G. Let $\mathrm{Syl}_p(G)$ be the set of Sylow p-subgroups of G.

Remark 5.2. If G is a finite group of order $p^k m$ with p prime and p not dividing m, then a Sylow p-subgroup of G has order at most p^k, by Theorem 4.46.

Problem 5.3. Find a Sylow 5-subgroup of S_5.

Problem 5.4. Find a Sylow 2-subgroup of S_4. *[Hint: the maximum possible cardinality is* 8*. Do you know of a group with* 8 *elements that acts on a set of size* 4*?]*

5.2. **Sylow's Theorem.**

Theorem 5.5. *Let p be a prime. If $P \in \mathrm{Syl}_p(G)$, then $gPg^{-1} \in \mathrm{Syl}_p(G)$ for all $g \in G$, so G acts on $\mathrm{Syl}_p(G)$ by conjugation.*

Theorem 5.6. *Let p be a prime. If P is a p-subgroup of a group G and Q is a p-subgroup of $N_G(P)$, then QP is a p-subgroup of $N_G(P)$.* [Hint: first show that QP/P is a p-group.]

Theorem 5.7. *Let p be a prime, and let P be a Sylow p-subgroup of a group G. If P is normal in G, then P is the only Sylow p-subgroup of G, and consequently, P is always the unique Sylow p-subgroup of $N_G(P)$.*

Theorem 5.8 (Sylow's Theorem - part 1)**.** *If G is a finite group and p is a prime dividing $|G|$, then any two Sylow p-subgroups of G are conjugate, and further, $|\mathrm{Syl}_p(G)| \equiv 1$ modulo p.*
[Hint: let $\mathcal{O}$ be an orbit of G acting on $\mathrm{Syl}_p(G)$ by conjugation. The goal is to show $\mathcal{O} = \mathrm{Syl}_p(G)$ and $|\mathcal{O}| \equiv 1 \bmod p$. Choose $P \in \mathcal{O}$, and towards a contradiction, assume that $Q \in \mathrm{Syl}_p(G)$ with $Q \notin \mathcal{O}$. Now, the key is to consider how P and Q act on $\mathcal{O}$ (by conjugation).

1. Show that the only subgroup in $\mathcal{O}$ that P fixes, i.e. normalizes, is P itself. Conclude that $|\mathcal{O}| \equiv 1$ modulo p.
2. Show that Q fixes nothing in $\mathcal{O}$. Conclude from this that $|\mathcal{O}| \equiv 0$ modulo p.

The previous theorem and Theorem 4.40 are very relevant.]

Theorem 5.9 (Sylow's Theorem - part 2)**.** *If G is a finite group and $|G| = mp^k$ with p prime and p not dividing m, then $|P| = p^k$ for every $P \in \mathrm{Syl}_p(G)$.*
[Hint: use part 1 of Sylow's Theorem and the Orbit-Stabilizer Theorem to show $|N_G(P)| =$

$m'p^k$ for some m'. Now, towards a contradiction, assume that $|P| = p^\ell$ with $\ell < k$, and consider the quotient group $N_G(P)/P$. Show that $N_G(P)/P$ must have an element of order p and use this find a contradiction.]

5.3. **Applications of Sylow's Theorem.**

Remark 5.10. Since all Sylow p-subgroups of a finite group are conjugate, a finite group has a normal Sylow p-subgroup if and only if it has a unique one. Thus, the condition "$|\operatorname{Syl}_p(G)| \equiv 1$ modulo p" can be helpful in determining if a group has a normal Sylow subgroup or not. And one should always remember that $|\operatorname{Syl}_p(G)| = |G : N_G(P)|$ by the Orbit-Stabilizer Theorem, so in particular, $|\operatorname{Syl}_p(G)|$ is always coprime to p.

Theorem 5.11. *If G is a group of order mp^k with p prime and $m < p$, then G has a normal Sylow p-subgroup.*

Theorem 5.12. *If G is a group of order pqr where p, q, and r are prime with $p < q < r$, then some Sylow subgroup of G is normal.* [Hint: the following counting technique often works well when the largest prime divisors of $|G|$ only occur to the first power (make sure you see when you use this). The rough idea is that if no Sylow subgroup of G is normal, then G will have too many Sylow subgroups and, in turn, too many elements. Assume the theorem is false. First count the number of Sylow r-subgroups, and use this to count the number of elements of G of order r. Now estimate (it will be hard to precisely count) the number of Sylow q-subgroups, and use this to estimate the number of elements of G of order q. Finally, compare the sum of these with the order of G.]

The End