Music, Math & Me: From Pythagorean Tuning to Digital Synthesis

E. Carmi D. Zeigler

California State University, Sacramento, CA

Abstract

Mathematically, audio signals are modelled as periodic curves, or waves, in the time domain. According to Fourier's Theorem, any periodic wave can be decomposed as a sum of sinusoidal waves. Applied to music, every instrument has a signature sound characterized by the frequencies of the sound waves produced when played. We used the Fast Fourier Transform (FFT) to extract and manipulate audio signal properties (amplitude, frequency and wave envelope) of a tenor saxophone. Ring modulation was also applied to manipulate a square wave signal.

Motivation

Equipped with the complete frequency profile of an instrument, a similar sound may be electronically recreated using sinusoidal waves. Knowledge of the sound profile of an instrument may also enhance sound quality. Pythagoras developed a structure of frequency ratios which are now referred to as musical scales. For reasons unknown, many modern musicians are unaware of Pythagorean Tuning and have been using equal-tempered tuning. We can compare multiples of frequencies that match the Pythagorean ratios with each other, as well as randomly selected frequencies to determine relative dissonance/consonance.

Components of Sound

Sound is characterized by:

- **Frequency (pitch)**
- **Duration (length)**
- **Amplitude (volume)**
- **Tone (harmonic content)**
- **Envelope (ADSR)**

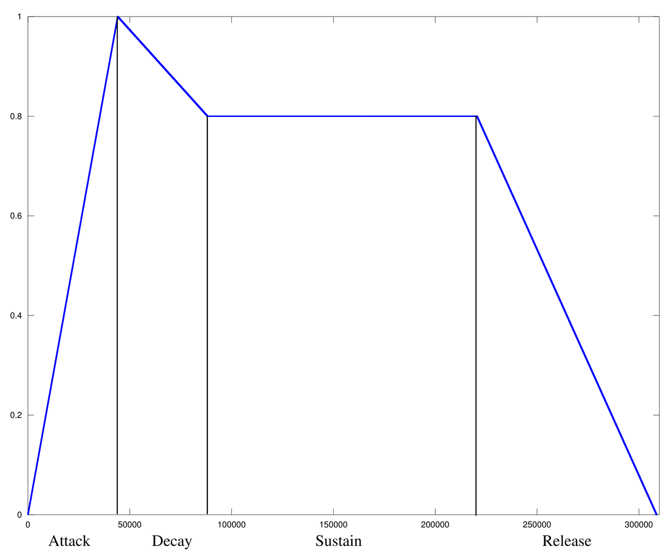


Figure: An envelope of sound is composed of a sound's attack, sustain, decay, and release.

Fast Fourier Transform (FFT)

The discrete Fourier transform (DFT) is used to analyze the frequency spectrum of a finite collection of sampled data. For an original 2π-periodic signal, $f(x)$, the sampled signal is $\mathbf{f} = (f_1\, f_2\, \cdots\, f_n)^T \in \mathbb{R}^n$. The transformed signal, $\hat{\mathbf{f}}$, containing frequency data is calculated as

$$\hat{\mathbf{f}} = \begin{pmatrix} 1 & \omega & \cdots & \omega^{(n-1)} \\ 1 & \omega^2 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix} \mathbf{f}$$

where

$$\omega = \exp\left(-\frac{2\pi i}{N}\right) = \cos\left(-\frac{2\pi}{N}\right) + i \sin\left(-\frac{2\pi}{N}\right)$$

The components of $\hat{\mathbf{f}}$ give a resolution of the sampled signal, $\mathbf{f}$, into it's harmonics [1]. The FFT is an optimized algorithm for computing the DFT that produces exactly the same result as evaluating the DFT definition directly at a much faster rate.

Pythagorean Tuning

Modern scales used in music are based on specific ratios compared to the first note, similar to frequency ratios.

Scale	Ratio (Freq.)	Interval Name
1	1	unison
2	9/8	major second
3	81/64	major third
4	4/3	perfect fourth
5	3/2	perfect fifth
6	27/16	major sixth
7	243/128	major seventh
8	2	octave

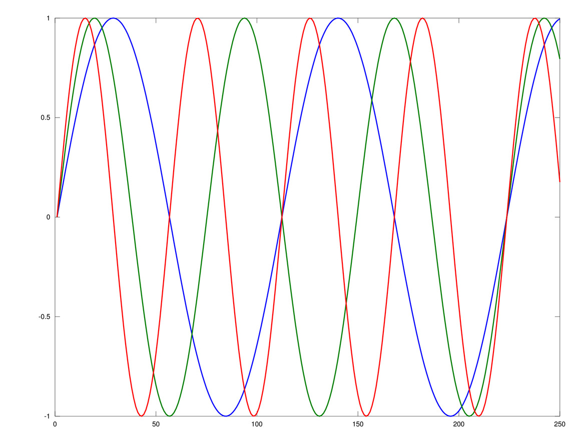


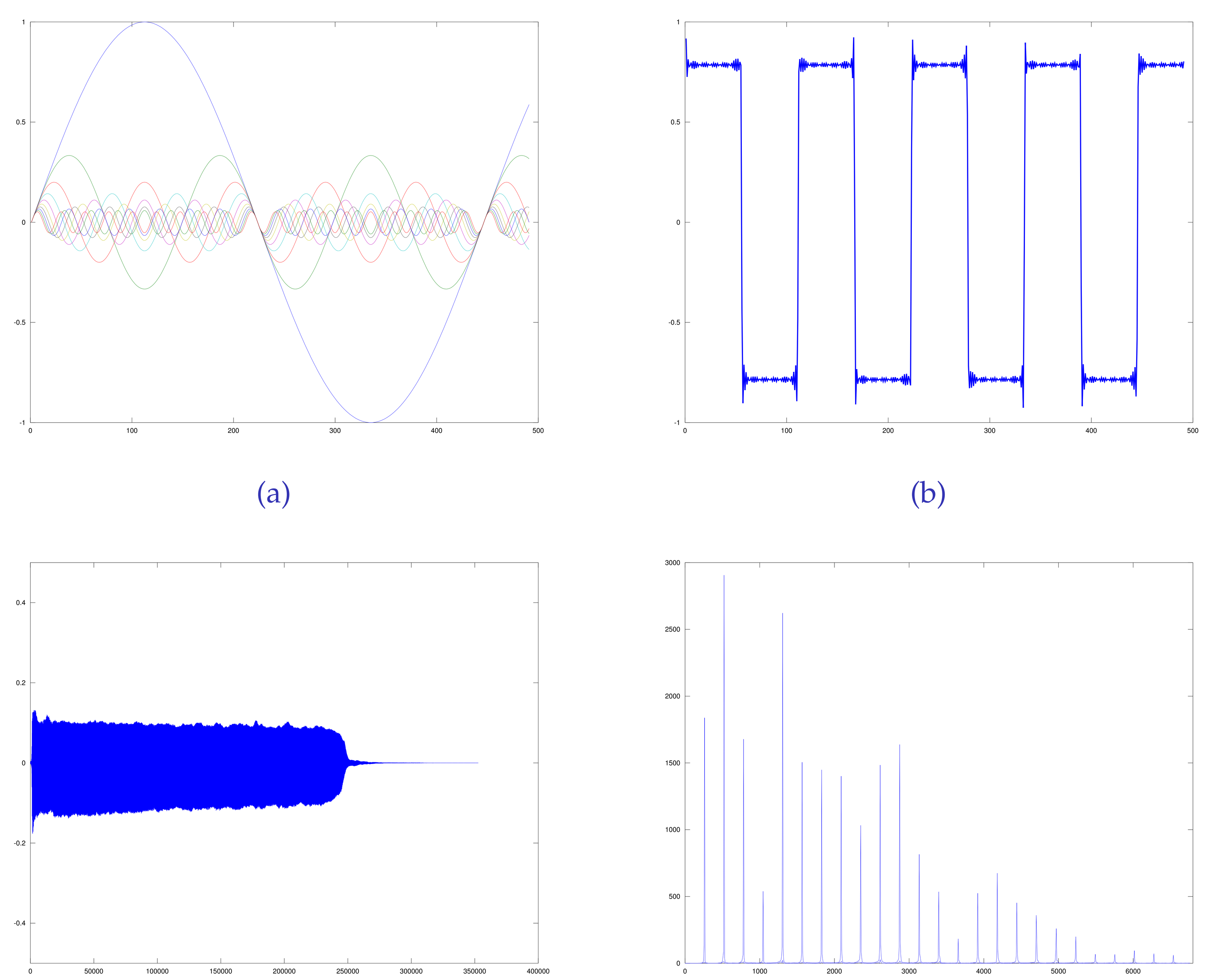
Figure: Frequency ratios for a single sinusoidal wave; $\times 1$ (red), $\times 3/2$ (green), $\times 2$ (blue)

Square Wave & Tenor Sax Harmonics

A square wave is composed of a sum of odd harmonics ($n = 1, 3, 5, 7, \ldots$) with each successive sine wave amplitude decreasing by a factor of $(1/n)$

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \sin x + \frac{1}{3}\sin 3x + \cdots$$

Using the FFT, the sampled signal of a tenor saxophone may be decomposed into it's unique spectrum of frequency ratios.

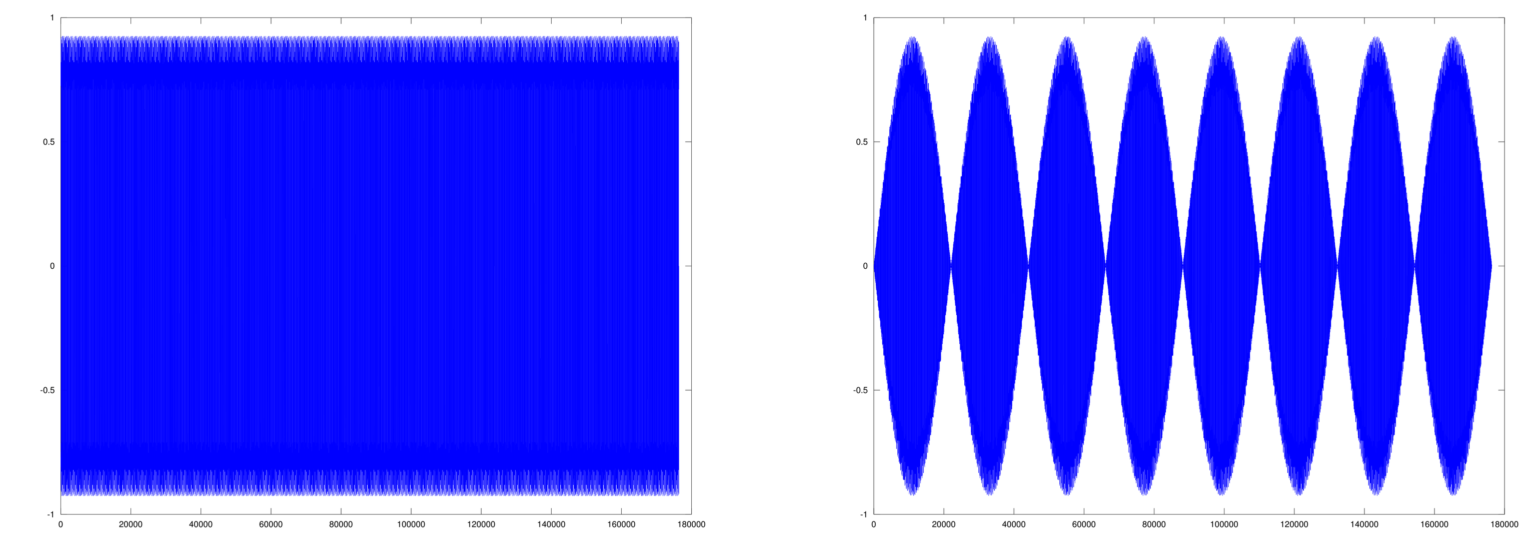


(a) (b) (c) (d)

Figure: (a) Individual odd harmonic waves; (b) summation of odd harmonic waves to form a square wave; (c) sampled signal of a tenor saxophone playing a single note; (d) frequency distribution of tenor sax found by applying the FFT to the audio signal in (c).

Ring Modulation

Ring modulation is a type of amplitude modulation where the input levels of the carrier and modulation signals are balanced such that the only output are the sum and difference frequencies. The resulting frequencies are almost guaranteed to not be harmonically related, and ring modulation is often used to simulate the sounds of tuned percussion instruments that produce inharmonic frequency spectra, such as bells and chimes. In this example, the carrier wave is a square wave of 396 Hz. When the modulating wave is a 1 Hz sine wave, the effect produced is similar to the carrier wave's volume changing sinusoidally between the maximum volume and no volume. When the modulating wave is of a higher frequency or a different waveform, the output has more aliasing (distortion) if the modulating wave's fundamental frequency is $\times 9/8$ (major second) from the carrier, but there is less distortion if the modulating wave is $\times 3/2$ (P5) or $\times 2$ (octave) from the carrier.



(a) (b)

Figure: (a) Carrier square wave of 396 Hz; (b) The effect produced after applying a modulating wave sine wave of 1 Hz.

Conclusion

Digital synthesis is a common application in creating or modulating sound for music. The FFT is incredibly useful for analyzing the frequency spectrum of a signal. These techniques are not restricted to audio signals and can also be applied in analyzing waves of light got radio communication or infrared imaging.

Future Work

- Create a phaser effect by implementing an all-pass filter with the z-transform.
- Create a digital delay, which shifts a sequence by a desired amount of samples.
- Create a flanger effect by combining the phaser and delay. The net sound effect imitates a Doppler shift.

Bibliography

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http://http://webpages.csus.edu/ zeigler

zeigler at csus dot edu