Modeling Steady-State Thermohaline Groundwater Flow Using Radial Basis Functions

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ABSTRACT

A meshless method is designed for modelling double-diffusive thermohaline groundwater flow in an aquifer. The algorithm uses the Kansa collocation method with Gaussian radial basis functions (RBFs). Numerical results are presented for the steady-state case with symmetric domain.

MOTIVATION

Flows with two sources of buoyancy are of great interest with respect to contaminant transport in groundwater. However, the second source of buoyancy dramatically complicates the dynamics of heat and mass transfer. Meshless methods have shown to be an effective approach to constructing numerical simulations of the fluid flow. Using RBFs is appealing because of their high algorithmic simplicity, and independence of dimension and coordinate system. To our best knowledge, modelling double-diffusive ground water flow using Kansa's method has not been attempted.

GOVERNING SYSTEM OF EQUATIONS

A two-dimensional equivalent porous medium that is homogeneous and isotropic is considered. Assuming an incompressible fluid with no sources or sinks, convection in porous media is described by the coupled, nonlinear system of equations

$$\nabla \cdot \mathbf{q} = 0$$
$$\mathbf{q} = -\nabla p + R\rho\nabla f$$
$$\rho = 1 - \alpha T + \beta C \qquad (1)$$
$$\nabla T \cdot \mathbf{q} = \triangle T$$
$$L\nabla C \cdot \mathbf{q} = \triangle C$$

where p is the pressure field, R a parameter based upon the Rayleigh number, f is a body potential, ρ is the density, T is the temperature, C is the concentration, and L is the porosity. The coefficients of thermal and chemical expansivity are α and β, respectively. By incompressibility, there exists a stream function, $\psi(x, z, t)$, such that $\mathbf{q} = (\psi_{,z}, 0, -\psi_{,x})^T$. Aside from specific configurations, the recursive and nonlinear behavior of the equations precludes an analytic solution. Hence, numerical approximations are the only means towards understanding the behavior of the fluid flow.

RADIAL BASIS FUNCTIONS

RBFs are members of the Beppo-Levi space of distributions on $\mathbb{R}^3$ with square integrable second derivatives, $BL^{(2)}(\mathbb{R}^3)$. The space $BL^{(2)}(\mathbb{R}^3)$ is equipped with the rotation invariant semi-norm defined as

$$\|s\| = \int_{\mathbb{R}^3} (\triangle s)^2 d\mathbf{x}$$

for any $s \in BL^{(2)}(\mathbb{R}^3)$ where $\triangle$ is the Laplacian operator. This semi-norm is a measure of the energy or "smoothness" of functions: functions with a small semi-norm are smoother than those with a large semi-norm. Duchon [1] showed that the smoothest interpolant, has the simple form

$$s(\mathbf{x}) = p(\mathbf{x}) + \sum_{i=1}^{N} c_i\phi(|\mathbf{x} - \mathbf{x}_i|)$$

where p is a polynomial of low degree and the basic function ϕ is a real valued function on $[0, \infty)$, usually unbounded and of non-compact support. In this context, the points, $\mathbf{x}_i$, are referred to as the centers of the RBF. For our model, we used the Gaussian RBF

$$\phi(r) = \exp(-cr^2)$$

KANSA'S METHOD

Consider the differential equation

$$Fu = f(\mathbf{x}), \ \forall \mathbf{x} \in \Omega$$
$$Gu = g(\mathbf{x}), \ \forall \mathbf{x} \in \partial\Omega \qquad (2)$$

for a domain $\Omega \subseteq \mathbb{R}^3$. Kansa's method [2] relies upon using the N centers for the basis functions, $\{\mathbf{x}_i\}_{i=1}^N$, as collocation points. Some of the centers, $\{\mathbf{x}_i\}_{i=1}^{N_I}$ are interior points (in Ω) and the remainder, $\{\mathbf{x}_i\}_{i=N_I+1}^N$, are boundary points. The approximate solution can be expressed as a linear combination of the radial basis functions:

$$u(\mathbf{x}) = \sum_{j=1}^{N} u_j\phi(|\mathbf{x} - \mathbf{x}_j|) \qquad (3)$$

Substituting this expansion into the equations (2) and matching known values at all of the collocation points, we obtain the system of equations

$$\sum_{j=1}^{N} u_j(F\phi)(|\mathbf{x}_i - \mathbf{x}_j|) = f(\mathbf{x}_i), \ i = 1, 2, \ldots, N_I$$
$$\sum_{j=1}^{N} u_j(G\phi)(|\mathbf{x}_i - \mathbf{x}_j|) = g(\mathbf{x}_i), \ i = N_I + 1, \ldots, N$$

If F or G is nonlinear, then a nonlinear solver is required to determine the coefficients. If F and G are linear, then the resulting coefficient matrix is positive definite.

PROBLEM FORMULATION

The computational domain consists of a unit box with a hot bottom, cold top and adiabatic sides. All walls are impermeable to flow and the fluid is initially cold and motionless. By spatial symmetry, the computational domain is simply a unit square with a corner at the origin. The boundary conditions are based upon the "salted from below" configuration ([3]).
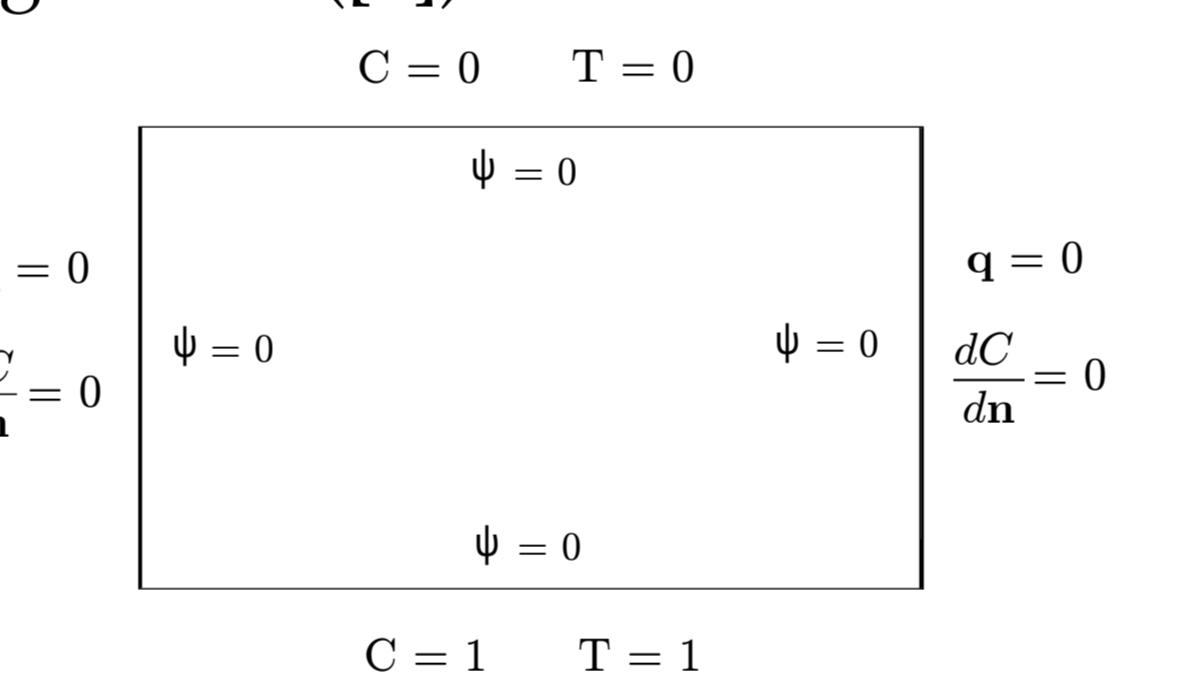
Figure: "Salted from below" boundary conditions: the top and bottom walls are kept at a constant composition and no salt is allowed to leave the box through the side walls.

The temperature and concentration fields are approximated as in (3) with five center/collocation points shown in Figure 2.
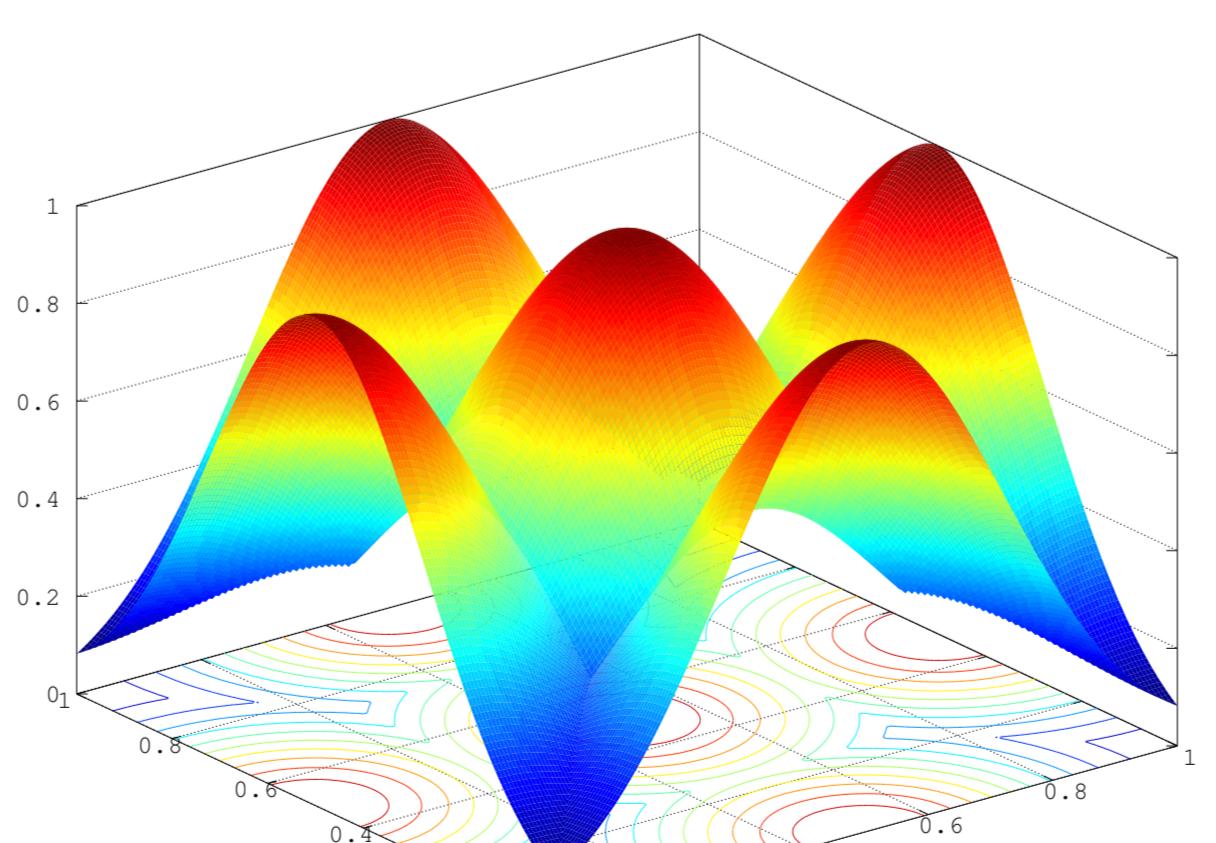

Figure: The location of the five collocation/center points: one along each edge and a single center located in the domain interior.

NUMERICAL RESULTS

Using the parameter values $\alpha = 3$, $\beta = 2$, $L = 1$ and $R = 1$, we obtained results comparable to Rosenberg and Spera [3]. The flow consists of a simultaneous rising hot fluid with a high solute concentration and a sinking colder fluid with lower solute concentration.


(a)

(b)
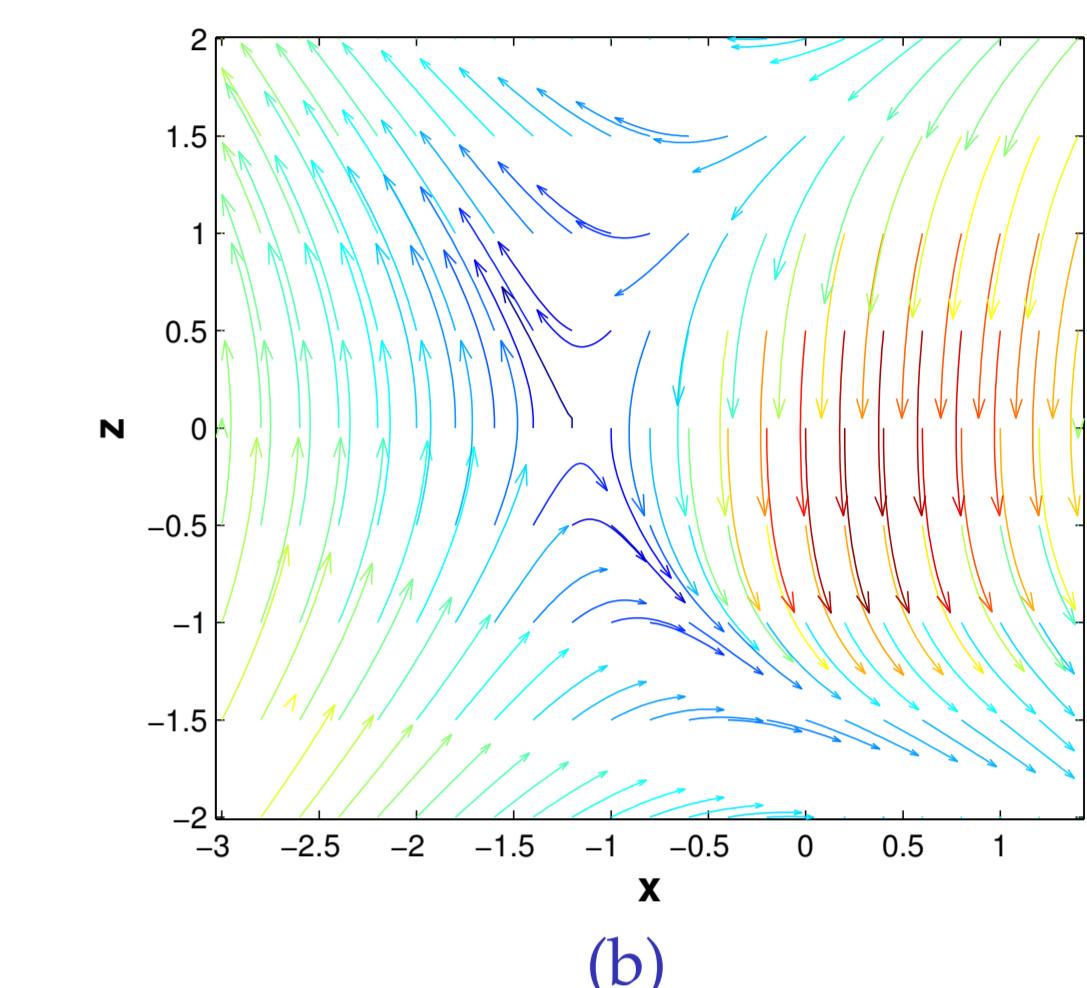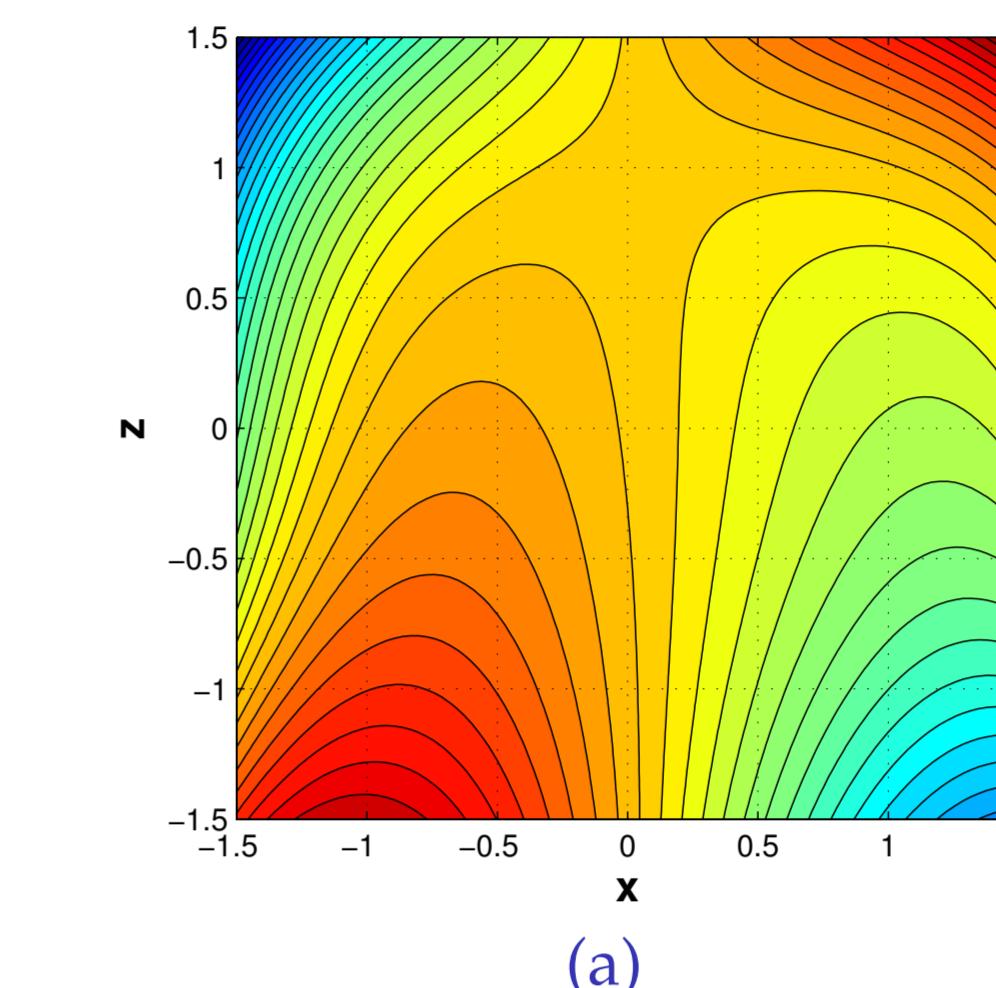
Figure: (a): Plot of the stream function, ψ, with fluid flowing from high potential (red) to low (blue). (b): Plot of the specific discharge, $\mathbf{q}$, with high (red) and low (blue) velocity flow indicated.

The modest number of data points required allowed this simulation to be completed on a laptop computer within a few minutes.

CONCLUSION

Kansa's method of collocation using RBFs is a versatile and robust tool for constructing approximate solutions to a wide variety of PDEs. In addition to the simplicity of formulating the corresponding linear system, collocation with RBFs is mesh-free and is well suited for problems with moving interfaces (e.g. dynamic crack growth, plastic deformation). For double-diffusive flow problems, our model yielded results comparable to existing models but with fewer collocation points.

FUTURE WORK

Our model utilized a set of evenly-spaced center/collocation points within the domain. Of the vast number of questions, there are two in particular we wish to investigate:

▸ Can sufficient accuracy be achieved with an arbitrary set of center/collocation points?
▸ Assuming a smooth extension of physical properties, can centers/collocation points be located outside of the domain?

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