Buoyancy-Driven Groundwater Flow as a Non-Chaotic System

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Abstract

Continuum models for the natural convection of a fluid through a porous media with two buoyancy sources are mathematically described as a system of coupled nonlinear differential equations. Because of the intractability of the governing equations, descriptions of the behavior of the system have been obtained exclusively through numerical investigations. A ubiquitous question when constructing numerical models is the accuracy of the model. Specifically, is the exhibited behavior actual physical phenomena or a numerical artifact? To resolve this question, a rigorous analysis of the system is required. A goal of our project is to establish the integrability of the system of differential equations describing groundwater flow where the behavior of the system is described using the generalized Darcy Law. This will be accomplished through the construction of a manifold on whose level surfaces all flow paths are confined. The argument presented relies upon techniques used to analyze dynamical systems and are independent of the boundary and initial conditions.

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Introduction

Buoyancy-driven fluid flow in porous media can occur in the presence of large temperature or concentration gradients. In such cases, the flow is strongly coupled to heat and solute transport and the system is described by a coupled nonlinear system of partial differential equations. Because they are nonlinear and recursive, the behavior of the flow and transport equations are typically investigated numerically. Some numerical simulations of free convection in porous media have produced flow and transport fields that display erratic behavior that has been interpreted as chaos.

When numerical models of nonlinear systems display chaotic behavior it raises a vexing problem as to whether the chaotic behavior is introduced by the numerical calculations or is inherent in the physical system. In our project, we present an analytic argument showing that flow in porous media cannot be chaotic even in the presence of arbitrarily large buoyancy forces. Our proof consists of showing the system is completely integrable. Through the Poincare-Bendixson theorem, such systems are known to be non-chaotic.

Flow in Porous Media

Fluid flow through an isotropic, inhomogeneous porous medium is described in nondimensional form by

$$\frac{d\mathbf{x}}{dt} = \mathbf{q}(t, \mathbf{x}(t))$$

Darcy's Law describes the specific discharge function

$$\mathbf{q} = -\nabla p + R\rho\nabla f$$

where R is a parameter related to the Rayleigh number, and f is a sufficiently smooth body potential. Fluid density is linearly dependent on temperature and solute concentration

$$\rho = 1 - \alpha T + \beta C$$

Fluid incompressibility yields

$$\mathrm{div}\ (\rho\mathbf{q}) = 0$$

Temperature and concentration are defined as

$$\begin{aligned} \frac{\partial T}{\partial t} + \nabla T \cdot \mathbf{q} &= \triangle T \\ \frac{\partial C}{\partial t} + \frac{1}{\phi}\nabla C \cdot \mathbf{q} &= \triangle C \end{aligned}$$

Lamb Surface

Using a variation of Hubbert's Potential

$$K = R\rho \qquad H = f + \int_{p_0}^{p} \frac{dp}{R\rho}$$

the specific discharge may be expressed as

$$q = -K\nabla H$$

The Lamb surface is a two-dimensional manifold containing the specific discharge and the corresponding vorticity.

$$\mathcal{H}(\mathbf{x}) = \frac{\partial \mathcal{H}}{\partial K}\int_{\mathbf{x}_0}^{\mathbf{x}} \left(-Kq^{-2}\right)\, \ell \cdot d\mathbf{x}$$

where

$$\ell = (\nabla K \times \nabla H) \times \mathbf{q}$$

Setting

$$\lambda = -Kq^{-2}\,\frac{\partial \mathcal{H}}{\partial K}$$

the coefficient is found by solving the differential equation

$$\frac{d\lambda}{dt} + \left[\frac{\partial}{\partial H}\left(\frac{dH}{dt}\right) + \frac{\partial}{\partial K}\left(\frac{dK}{dt}\right)\right]\lambda = 0$$

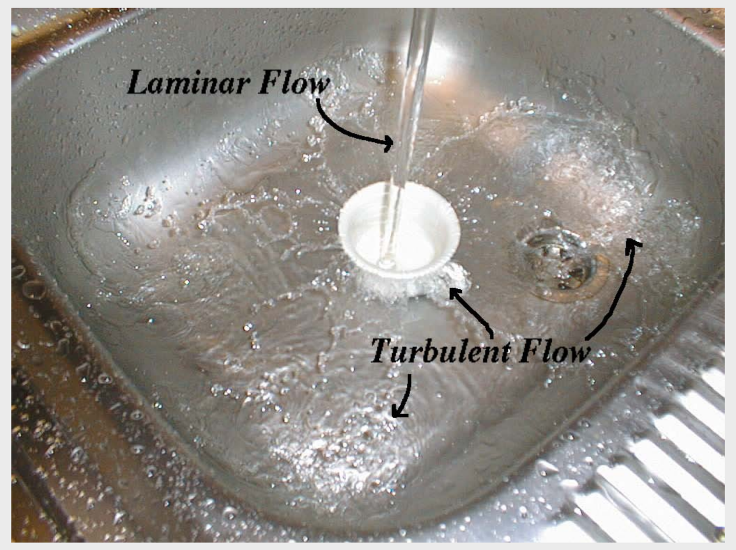

Figure 1. Turbulent (chaotic) flow is characterized by recirculation, eddies, and apparent randomness. Flow in which turbulence is not exhibited is called laminar. Note, however, that not all chaotic flow is turbulent.

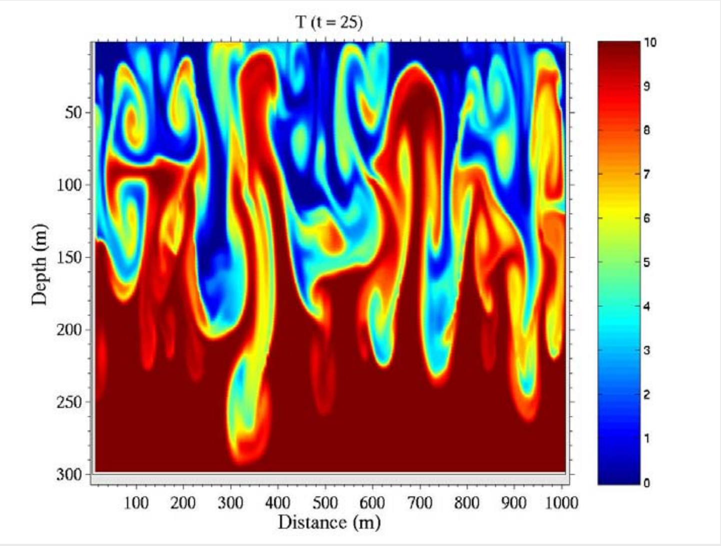

Figure 2. Fingering pattern in double-diffusive flow just prior to onset of convection. Heat causes the fluid to rise. Cooler, less dense fluid sinks due to gravity. Simultaneously, more saline fluid has greater density and sinks. Density and temperature differences induce convection.

Integrability & Stability

Poincare-Bendixson states that if a trajectory enters and does not leave a closed and bounded region of phase space which contains no equilibria, then the trajectory must approach a periodic orbit as $t \to \infty$.

Suppose the fluid is slightly inhomogeneous as described by a perturbation expansion of the Reynold's number

$$R = \frac{1}{1-r} = \sum_{i=0}^{\infty} r^i, \quad 0 < r < 1$$

Then we can construct approximate Lamb surfaces such that

$$\mathcal{H} = \sum_{i=0}^{\infty} r^i \mathcal{H}_i$$

The series is convergent for all r and absolutely convergent when

$$\frac{2}{3} < R < 2$$

Conclusions & Future Work

- Double-diffusive flows are confined to manifolds described by the Lamb surface.
- Yet more evidence that mathematics is superior and mathematicians are the biggest and baddest.
- Is zero helicity a necessary condition for integrable flows?
- Better numerical models are needed (especially if the Rayleigh number is large!)

References

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